Equation (7) is solved by collocation using values of G and α_{i_ϵ} obtained by numerical integration. Once the A_n are determined, the wing lift coefficient can be obtained in the form

$$C_L = C_{L_\alpha}\alpha + C_{L_\epsilon}\epsilon \tag{10}$$

Results and Discussion

The lifting-surface[2] and vortex-lattice[3] solutions are given for rectangular wings with $A=1$, 2, and 4 and agree well with each other and experiments. For comparison purposes, a vortex-lattice method was developed. For $A=4$, these vortex-lattice results agree well with those in Refs. 2 and 3 and will be used in the figures. To handle the problem with thickness, the downwash at the control points included the contribution from the distributed source images obtained by numerical integration.

The slope of the wing lift coefficient vs angle-of-attack curve C_{L_α} is plotted vs $(2h/b)$ in Figs. 1 and 2 for $A=4$ and 6, respectively. $(C_{L_\alpha})_\infty$ is the result for $h \to \infty$. The lifting-line results are seen to agree well with the vortex-lattice results, with better agreement at the larger aspect ratio as expected. The lifting-line results can be extended to smaller values of $(2h/b)$ if the exact two-dimensional value of a_0 determined by Havelock[6] is used.

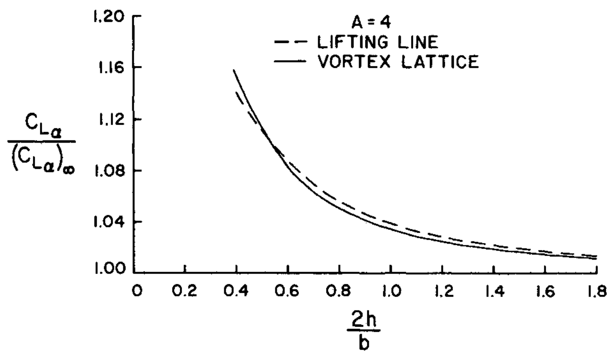

Fig. 1 Lift curve slope for flat rectangular wing of aspect ratio 4.

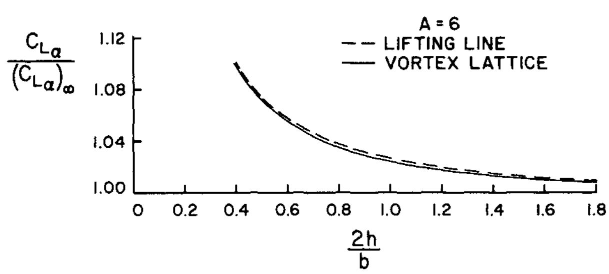

Fig. 2 Lift curve slope for flat rectangular wing of aspect ratio 6.

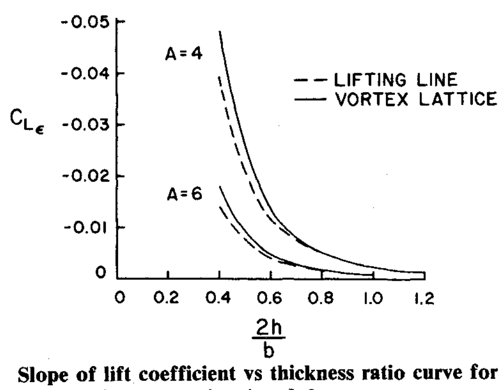

Fig. 3 Slope of lift coefficient vs thickness ratio curve for flat rectangular wings of aspect ratios 4 and 6.

The slope of the wing lift coefficient vs thickness ratio curve C_{L_ϵ} is plotted vs $(2h/b)$ in Fig. 3. The results of the lifting-line and vortex-lattice calculations again are seen to agree reasonably well, and thickness is seen to decrease the lift.

As a final point of interest, the lifting-surface, vortex-lattice, and lifting-line theories discussed here all assume a linear relationship between lift and angle of attack. For the two-dimensional, zero-thickness ground effect problem, an expansion of the lift coefficient of Havelock[6] in both c/h and α, keeping terms to $0(\alpha^2)$, yields

$$C_\ell = 2\pi\alpha\left\{1 - \frac{\alpha}{2}\left(\frac{c}{h}\right) + \frac{1}{16}\left(\frac{c}{h}\right)^2 + \ldots\right\} \tag{11}$$

For smaller values of h/c, it is seen that the presence of the ground introduces a nonlinearity in α much stronger than in the infinite fluid problem. It is therefore expected that the wing in ground effect results of Refs. 2 and 3 and those presented here are valid for a more restricted range of angle of attack than is usually appropriate for these theories in the absence of the gound plane.

Acknowledgment

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Integration of Singular Functions Associated with Lifting Surface Theory

Becker van Niekerk*
Stanford University, Stanford, California

Nomenclature

C,c = constants
E = error term in integration rules
F = function
f,g = arbitrary functions
H = quadrature weights
I = integral
i_n = index from 1 to n

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*Research Assistant, Department of Aeronautics and Astronautics. Student Member AIAA.

K = kernel function
n = upper limit, index
x,ξ = chordwise coordinates on lifting surface planform
x_0 = $(x-\xi)$
y,η = spanwise coordinates
y_0 = $(y-\eta)$

Introduction

THE application of linearized potential theory to obtain the flow past, and airloads on, almost planar thin wings in subsonic flow leads to various singular integral equations.[1] The kernel functions of the integral equations depend on whether it is steady or unsteady flow, what is chosen as the unknown, what is to be calculated, etc. In particular, the integral equation relating the downwash on the wing to the difference in pressure between the upper and lower surfaces exhibits a $1/y_0^2$ singularity for both steady and unsteady flow. y_0 is the spanwise distance between a sending doublet at (ξ,η) and a control point at (x,y). The kernel must be integrated over the span. Even though it is divergent in the classical sense, this form is the result of repeated differentiations of a nonsingular kernel, such that its integral is well defined. Mangler[2] has given rules to obtain the "finite part" of these double pole singular integrals.

To integrate this strong singularity numerically, it has been standard practice in kernel function methods to subtract it out analytically.[3] In general, it is then necessary to evaluate derivatives of the kernel function as well, a process that involves a substantial amount of algebra. If a different form of the kernel is to be used, all of this algebra must be repeated. Hence, extension of the kernel function approach to related problems, such as those including the effects of the wind tunnel walls or carrying out time-accurate transient flow computations, is quite involved.

In an April 1985 seminar at Stanford University, Giovanni Monegato[4] first suggested to the author the possibility of using direct numerical integration to evaluate the "finite part" of double-pole integrals. This approach seems to have been overlooked by researchers in aerodynamics. This Note outlines a simple quadrature rule that can be directly applied in subsonic aerodynamic problems.

If the rule presented below is implemented in a lifting surface code, the only requirement is that, for any given kernel function,

$$\lim_{y_0\to 0} y_0^2 K(x_0,y_0)$$

be known. This limit is, in general, not difficult to find. Once a basic lifting surface code has been written, it is easy to adapt it to other problems. In a practical implementation, gains in numerical efficiency may be realized. The main advantage, however, is that applying kernel function methods becomes conceptually as simple as the vortex lattice or other panel methods, because the careful identification and treatment of the singularities is unnecessary.

Finite Part of a Double-Pole Singular Integral

Define the finite part integral as

$$I(y) = \int_{-1}^{1} \frac{f(\eta)}{(\eta-y)^2}\mathrm{d}\eta \tag{1}$$

It is understood that the "finite part" of the integral is to be found in the sense of Mangler.[2] $f(\eta)$ is not permitted to have any singularities at $\eta=0$.

For the treatment of lifting surface integral equations, Gauss-type integration rules are a very efficient scheme of numerical quadrature. An n-point Gauss formula will integrate polynomials up to degree $(2n-1)$ exactly. The superior convergence of Gauss formulas can be attributed to the fact that the abscissas and weights are determined uniquely and in an optimal sense.[5] Another feature of the Gauss procedure is that integration rules can be derived for various weighting functions that may appear in the integrand. Many such rules are available in the literature.[6]

Kutt[7] gave general Gauss formulas with $1/y^2$ as the weight function, but, unfortunately, the monomial integrations $\int y^n/y^2 \mathrm{d}y$ are not all positive. The result is that Kutt's weights can become complex numbers and the abscissas called for may lie outside the range of integration. In aerodynamic applications, this will lead to the evaluation of the integrand off the wing surface, which can cause problems.

There are many different interpretations of exactly what is meant by Eq. (1). One that we will find particularly useful is that

$$I(y) = \frac{\partial}{\partial y}\int_{-1}^{1} \frac{f(\eta)}{(\eta-y)}\mathrm{d}\eta \tag{2}$$

The Cauchy principal value integral in Eq. (2), for $y=0$, can be evaluated accurately with an even, symmetric Gauss-Legendre rule[8] for any function $f(\eta)$ that is regular at the point $\eta=0$. It has been suggested[9] that, for arbitrary y, it may be evaluated as

$$\int_{-1}^{1} \frac{f(\eta)}{(\eta-y)}\mathrm{d}\eta = \sum_{i_n=1}^{n} H_{i_n}\frac{f(\eta_{i_n})}{(\eta_{i_n}-y)} + F(y)f(y) + E \tag{3}$$

for $y\neq\eta_{i_n}$, $i_n=1, \ldots, n$. In Eq. (3), $F(y)$ is a function of y, E the error term, and H_{i_n} and η_{i_n} the weights and abscissas, respectively, for a Gauss-Legendre quadrature rule. Since a symmetric, even Gauss-Legendre rule gives the correct result, it must hold that $F(0)=0$ for any function $f(\eta)$ that is smooth at $\eta=y$. Using Eqs. (2) and (3), one sees that $I(0)$ may be evaluated as

$$I(0) = \sum_{i_n=1}^{n} H_{i_n}\frac{f(\eta_{i_n})}{\eta_{i_n}^2} + Cf(0) + E \tag{4}$$

where C is a real constant and must be determined such that, when $f(\eta)$ is a polynomial of order $(2n-1)$ or less, the "exact" answer for $I(0)$ is obtained. The "exact" answer is the value of the finite part of the integral evaluated in the Mangler sense. In particular,

$$\int_{-1}^{1} \frac{\mathrm{d}\eta}{\eta^2} = -2 \tag{5}$$

If C is determined such that Eq. (5) is satisfied, then all polynomials up to order $(2n-1)$ are also integrated exactly, because all the monomials $\eta, \eta^2, \ldots \eta^{2n-1}$ are zero at the origin. The quadrature then reduces to a standard Gauss-Legendre integration.

H_{i_n} and η_{i_n} are weights and abscissas for a Gauss integration with a unit weight function. C is then given by

$$C = -\sum_{i_n=1}^{n} \frac{H_{i_n}}{\eta_{i_n}^2} - 2 \tag{6}$$

and the final integration rule takes the form

$$I(0) = \sum_{i_n=1}^{n} H_{i_n}\frac{f(\eta_{i_n})}{\eta_{i_n}^2} + Cf(0) \tag{7}$$

The following properties of this rule should be noted:

1) An extra evaluation of the function $f(0)$ in Eq. (7) destroys the optimality of the original Gauss-Legendre quadrature.

2) It reduces to the standard Gauss-Legendre integration for functions of the form $\eta^2 g(\eta)$, with g nonsingular.

3) Only even Gauss-Legendre rules can be used.

4) The $1/\eta_{i_n}^2$ can be incorporated in the weights H_{i_n}.

Table 1 Accuracy of the integration rule for $I(0)$

$f(\eta)$	Exact	Two-point	Four-point
e^{η}	−0.9716595	−0.9719117	−0.9716595
$\cos^{-1}\eta$	−3.1415926	−3.1415926	−3.1415926
$\sqrt{1+\eta^2}$	−1.0656799	−1.0717967	−1.0657672
$(1-\eta^2)\log(1+\eta^2)$	0.2240980	0.383576	0.2259250

5) There is no numerical difficulty involved in obtaining the weights and abscissas because the standard Gauss-Legendre coefficients used can be found to great precision from tables or stable numerical computations.[6]

The effectiveness of this quadrature rule is demonstrated in Table 1, where $\int_{-1}^{1} f(\eta)/\eta^2 \mathrm{d}\eta$ for four different functions $f(\eta)$ is shown. The exact value is compared with two and four-point rules and it can be seen that the convergence is quite satisfactory.

Finite value integrals can be scaled and translated. A further useful property, which facilitates the computation of $I(y)$ for general limits of integration, is that

$$\int_a^b \frac{f(\eta)}{(\eta-y)^2}\mathrm{d}\eta = \int_a^{y-c} \frac{f(\eta)}{(\eta-y)^2}\mathrm{d}\eta + \int_{y-c}^{y+c} \frac{f(\eta)}{(\eta-y)^2} + \int_{y+c}^{b} \frac{f(\eta)}{(\eta-y)^2}\mathrm{d}\eta \tag{8}$$

with $y+c$ and $y-c$ within the limits of the original integral. This statement can be proved rigorously.[7] The finite part can then be evaluated using Eq. (7) after the appropriate scaling and translation, although care should be taken to obtain the nonsingular parts for $c \ll 1$ because of the large $1/\eta^2$ contribution. A special Gauss rule with $1/y^2$ weighting may be used in this case.

Conclusion

A special quadrature rule, which can be used for the direct numerical evaluation of the finite part of a double-pole singular integral, has been presented. Its application to subsonic aerodynamic problems greatly simplifies the algebra involved in lifting surface theories, a fact that makes it possible to use the kernel function approach for complex problems. The author is currently involved in applying this technique to compute time-accurate transient aerodynamic loads on wings.[10]

To date, no attempt has been made to investigate with full generality the convergence properties of the given quadrature rule.

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Quasi-One-Dimensional Gas/Particle Nozzle Flows with Shock

Magnar Førde*
Norwegian Institute of Technology
Trondheim, Norway

Nomenclature

A	= cross-sectional area, m^2
A_g	= cross-sectional area occupied by the gas phase, m^2
A_p	= total surface of the particles, m^2
b	= velocity of sound in gas/particle mixture, m/s
c	= velocity of sound in gas phase, m/s
c_D	= particle drag coefficient
c_p, c_v	= gas specific heat at constant pressure and volume, respectively, J/kg K
c_s	= particle heat capacity, J/kg K
d	= particle diameter, m
E_g, E_p	= internal energy per unit volume, J/m^3
f	= drag coefficient ratio
F	= interaction force between the phases
G_g, G_p	= mass flow, kg/s
h	= convective heat transfer coefficient, W/m^2k
H_g	= total enthalpy, J/kg
k	= thermal conductivity (gas), W/mk
M, M_p	= gas- and particle-phase Mach numbers, respectively
N	= number of particles per unit volume
Nu	= particle Nusselt number
p	= gas-phase pressure, N/m^2
Q	= heat flux per unit length, W/m^3
R	= gas constant, J/kg K
Re_p	= particle Reynolds number
S	= slip ratio
T_g, T_p	= temperature, K
T_r	= temperature ratio T_p/T_g
u_g, u_p	= velocities, m/s
x	= axial coordinate, m
α, α_p	= void fraction
β	= loading ratio, G_p/G_g
μ_g	= gas viscosity, Ns/m^2
κ	= specific heat ratio
ρ_g, ρ_p	= density, kg/m^3

Superscripts

(*)	= bulk conditions

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*Assistant Chief Engineer.